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# Birkhoff normal form of Hamiltonian systems and WKB-type formal solutions

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## 1 Introduction

As is illustrated by the computation of monodromy groups of second-order Fuchsian equations (cf. [AKT1]), the exact WKB analysis provides us with a powerful tool for studying global behavior of solutions of linear ordinary differential equations. To generalize such an analysis to nonlinear equations, T. Kawai (RIMS, Kyoto Univ.), T. Aoki (Kinki Univ.) and the author have developed the WKB theory for Painlevé equations with a large parameter in our series of articles ([KT1], [AKT2], [KT2]). (See [T1], [T2] also.) Although we have almost succeeded in analyzing the behavior of 2-parameter formal solutions constructed in [AKT2] near simple turning points (cf. [KT2]), their behavior near fixed singular points, which is also important for the global study of Painlevé equations, has not been clarified yet. The aim of this report is thus to consider the following problem: *How do our formal solutions behave near fixed regular-type singular points for Painlevé equations*?

In the case of second-order linear equations, the corresponding formal solutions are given by the WKB solutions and two typical methods are known for their construction: One is to transform equations in question into Riccati equations, and the other is to solve the so-called eiconal equation and transport equations. Between these two methods the first one is more effective to determine the behavior of WKB solutions near regular singular points. Now, to construct 2-parameter formal solutions of Painlevé equations with a large parameter, we have employed the multiple-scale analysis in [AKT2], that is, we have constructed them by solving some differential equations degree by degree. In this sense this construction corresponds to the second method for WKB solutions mentioned above and hence is not efficient to discuss the present problem. In this report we propose a new construction of 2-parameter formal solutions of Painlevé equations with a large parameter to investigate their behavior near fixed regular-type singular points.

The new construction of formal solutions we propose here is based on the work of Kimura [K] and its improvement by Takano [Tka1] (see [Tka2] also), where they respectively constructed a 2-parameter family of analytic solutions at each regular-type singular point of (ordinary) Painlevé equations. Making use of the well-known fact that Painlevé equations can be written in the form of Hamiltonian systems (which we call Painlevé Hamiltonian systems in this report), Kimura first established some reduction theorem for Hamiltonian systems to construct analytic solutions and later Takano modified his method to enlarge the domain of convergence of these analytic solutions. Their reduction theorem is closely related to the following "Birkhoff normal form" of Hamiltonian systems (cf. [B], [SM]).

**Birkhoff normal form** *Consider a Hamiltonian system*

$$dq/dt = \partial H/\partial p, \qquad dp/dt = -\partial H/\partial q \tag{1}$$

*with a Hamiltonian* $H = H(t, q, p)$. *If we can find a canonical transformation* $(q, p) \longrightarrow (\widetilde{q}, \widetilde{p})$ *which transforms the original system (1) to*

$$d\widetilde{q}/dt = \partial\widetilde{H}/\partial\widetilde{p}, \qquad d\widetilde{p}/dt = -\partial\widetilde{H}/\partial\widetilde{q} \tag{2}$$

*with*

$$\widetilde{H}(t, \widetilde{q}, \widetilde{p}) = \sum_{n\geq 0} \widetilde{h}_n(t)\,(\widetilde{q}\widetilde{p})^{n+1} \tag{3}$$

*(i.e.,* $\widetilde{H}$ *is a function of* $t$ *and the product* $\widetilde{q}\widetilde{p}$ *only), then the new system (2) is called Birkhoff normal form of (1).*

Roughly speaking, to construct 2-parameter formal solutions, we will revise their reduction theorem so that it may be adapted to Hamiltonian systems of singular perturbations and prove the existence of a canonical transformation which reduces the Painlevé Hamiltonian system to its "Birkhoff normal form" in a singular-perturbative manner. The existence of singular-perturbative reduction will be discussed in Section 3 and the behavior near fixed regular-type singular points of our 2-parameter formal solutions thus constructed will be investigated in Section 4. Before considering Painlevé Hamiltonian systems, in Section 2 we will study the relationship between this viewpoint and WKB solutions of second-order linear ordinary differential equations.

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## 2 Birkhoff normal form and WKB solutions of Schrödinger equations

In this section we discuss the construction of WKB solutions of 1-dimensional Schrödinger equations

$$\left(-\frac{d^2}{dx^2}+\eta^2 Q(x)\right)\psi=0 \qquad (\ \eta : \text{large parameter}) \tag{4}$$

from the viewpoint of reduction of Hamiltonian systems to their Birkhoff normal form. Let us begin by reviewing two well-known methods for the construction of WKB solutions.

The first method is to transform the unknown function $\psi$ of (4) into $S$ defined by

$$\psi=\exp\int^x S dx. \tag{5}$$

Then we readily verify that $S$ must satisfy the so-called Riccati equation:

$$S^2+\frac{dS}{dx}=\eta^2 Q(x). \tag{6}$$

This equation (6) has the following two formal power series solutions denoted by $S_\pm$:

$$\begin{aligned} S_\pm &= \pm\eta S_{-1}(x)+S_0(x)\pm\eta^{-1}S_1(x)+\cdots, \\ &= \pm S_{\text{odd}}+S_{\text{even}} \end{aligned} \tag{7}$$

where $S_{-1}(x)=\sqrt{Q(x)}$ and the other $S_j(x)$ ($j\geq 0$) are determined recursively. Note that the comparison of odd order terms (with respect to the power of $\eta$) of both sides of (6) entails

$$S_{\text{even}}=-\frac{1}{2}\frac{d}{dx}\log S_{\text{odd}}. \tag{8}$$

Substituting (7) and (8) into (5), we obtain the WKB solutions of (4) of the form

$$\psi_\pm=\frac{1}{\sqrt{S_{\text{odd}}}}\exp\int^x\left(\pm S_{\text{odd}}dx\right). \tag{9}$$

On the other hand, in the second method we seek for a solution of (4) in the following form:

$$\psi=\exp(\eta p(x))A(x), \qquad \text{where} \quad A(x)=a_0(x)+\eta^{-1}a_1(x)+\cdots. \tag{10}$$

In order that $\psi$ of the form (10) may be a solution of (4) $p(x)$ and $A(x)$ should satisfy

$$(11)\qquad \begin{cases} \left(\dfrac{dp}{dx}\right)^2 = Q(x), \\ \dfrac{d^2p}{dx^2}A + 2\dfrac{dp}{dx}\dfrac{dA}{dx} + \eta^{-1}\dfrac{d^2A}{dx^2} = 0. \end{cases}$$

Hence $dp/dx = \pm\sqrt{Q(x)}$ ("eiconal equation") and each coefficient $a_j(x)$ of $A(x)$ should be determined by the following differential equations ("transport equations") in a recursive manner:

$$(12)\qquad \left\{4Q(x)\frac{d}{dx} + Q'(x)\right\} a_j(x) = \mp 2\sqrt{Q(x)}a''_{j-1}(x) \qquad (j \geq 0).$$

(Here and in what follows $'$ denotes the differentiation with respect to $x$ and we conventionally define $a_{-1}(x) \equiv 0$.) In this way the eiconal equation and transport equations also determine the WKB solutions of (4) of the form (10). The solutions thus obtained are essentially the same with (9).

Let us now reconsider the construction of WKB solutions from the viewpoint of reduction of Hamiltonian systems. To do so, by putting $\varphi = \eta^{-1}d\psi/dx$ we rewrite the equation (4) in the following Hamiltonian form:

$$(13)\qquad d\psi/dx = \eta\partial H/\partial\varphi, \qquad d\varphi/dx = -\eta\partial H/\partial\psi$$

where

$$(14)\qquad H = H(x,\psi,\varphi) = \frac{1}{2}\varphi^2 - \frac{1}{2}Q(x)\psi^2.$$

This system (13) is a Hamiltonian system of singular perturbations. What we want to do is to transform (13) into its Birkhoff normal form by some canonical transformation $(\psi,\varphi) \longrightarrow (\widetilde{\psi},\widetilde{\varphi})$. In this case such a canonical transformation should be linear, i.e.,

$$(15)\qquad \begin{cases} \psi &= a(x)\widetilde{\psi} + b(x)\widetilde{\varphi} \\ \varphi &= c(x)\widetilde{\psi} + d(x)\widetilde{\varphi}, \end{cases}$$

and the Birkhoff normal form should be of the form

$$(16)\qquad d\widetilde{\psi}/dx = \eta\partial\widetilde{H}/\partial\widetilde{\varphi}, \qquad d\widetilde{\varphi}/dx = -\eta\partial\widetilde{H}/\partial\widetilde{\psi}$$

where

$$(17)\qquad \widetilde{H} = f(x)\widetilde{\psi}\widetilde{\varphi}.$$

(Here $a(x), \ldots, f(x)$ may depend on $\eta$ also). If we successfully find such a canonical transformation and a normal form, we automatically obtain solutions of the original equation (4) in the following way: The reduced system (16) are easily solved and

$$(18)\qquad \begin{cases} \tilde{\psi} &= \alpha \exp \eta \displaystyle\int^x f(x)dx \\ \tilde{\varphi} &= -\beta \exp\left(-\eta \displaystyle\int^x f(x)dx\right) \end{cases}$$

gives a solution of it. (Here $\alpha$ and $\beta$ denote free parameters and we have added minus sign $(-)$ in front of $\beta$ for the sake of convention.) Then, substitution of (18) into (15) produces the following solution of (4):

$$(19)\qquad \psi = \alpha a(x) \exp\left(\eta \int^x f(x)dx\right) - \beta b(x) \exp\left(-\eta \int^x f(x)dx\right).$$

Our problem is thus to find such a linear canonical transformation (15). Roughly speaking, we employ an inductive argument (with respect to the power of $\eta^{-1}$) to construct a canonical transformation. To illustrate our inductive argument, let us first consider the top degree part of the problem. Since the original Hamiltonian is given by (14), as the top degree part of the transformation we choose

$$(20)\qquad \begin{cases} \tilde{\psi} &= 2^{-1/2}Q(x)^{-1/4}\left(\sqrt{Q(x)}\psi + \varphi\right) \\ \tilde{\varphi} &= 2^{-1/2}Q(x)^{-1/4}\left(-\sqrt{Q(x)}\psi + \varphi\right), \end{cases}$$

that is,

$$(21)\qquad \begin{cases} \psi &= 2^{-1/2}Q(x)^{-1/4}\left(\tilde{\psi} - \tilde{\varphi}\right) \\ \varphi &= 2^{-1/2}Q(x)^{1/4}\left(\tilde{\psi} + \tilde{\varphi}\right). \end{cases}$$

Note that the factors $2^{-1/2}Q(x)^{-1/4}$ etc. are added so that the transformation becomes canonical. Then, by straightforward computations, we find that the system (13) is transformed into another Hamiltonian system with the Hamiltonian

$$(22)\qquad \widetilde{H} = \sqrt{Q(x)}\tilde{\psi}\tilde{\varphi} + \eta^{-1}\frac{Q'(x)}{8Q(x)}\left(\tilde{\psi}^2 - \tilde{\varphi}^2\right).$$

For the top degree part (22) is now of the required form, that is, its top degree part has the same structure with the Hamiltonian (17) of the Birkhoff normal form. Similarly, by adding appropriate degree $(-1)$ terms to the transformation (20) or (21) we could obtain a Hamiltonian system which is the Birkhoff normal form up to the degree $(-1)$, and this procedure could further be continued up to arbitrarily higher orders with respect to $\eta^{-1}$. However, to construct a canonical transformation in all orders, we here employ the following argument, which is conciser than the naive inductive argument explained above.

Let us assume that a transformation we are seeking for has the following form:

$$(23) \qquad \begin{cases} \psi &= a(x,\eta)\tilde{\psi} + b(x,\eta)\tilde{\varphi} \\ \varphi &= c(x,\eta)\tilde{\psi} + d(x,\eta)\tilde{\varphi}, \end{cases}$$

where $a(x,\eta)$ etc. are formal power series of $\eta^{-1}$. To guarantee that (23) is canonical, we suppose

$$(24) \qquad a(x,\eta)d(x,\eta) - b(x,\eta)c(x,\eta) = 1.$$

The transformation (23) is obtained also by using the following generating function $W(x,\tilde{\psi},\varphi)$:

$$(25) \qquad W(x,\tilde{\psi},\varphi) = -\frac{b}{2d}\varphi^2 + \frac{c}{2d}\tilde{\psi}^2 - \frac{1}{d}\tilde{\psi}\varphi,$$

in other words, (23) is equivalent to

$$(26) \qquad \psi = -\partial W/\partial\varphi, \qquad \tilde{\varphi} = -\partial W/\partial\tilde{\psi}.$$

The relation between the original Hamiltonian and the transformed one is described also in terms of the generating function $W$ as follows:

$$\begin{aligned}
(27)\ \widetilde{H} &= H(x,\psi(\tilde{\psi},\tilde{\varphi}),\varphi(\tilde{\psi},\tilde{\varphi})) + \eta^{-1}\frac{\partial W}{\partial x}(x,\tilde{\psi},\varphi(\tilde{\psi},\tilde{\varphi})) \\
&= \frac{1}{2}\left(c\tilde{\psi}+d\tilde{\varphi}\right)^2 - \frac{1}{2}Q(x)\left(a\tilde{\psi}+b\tilde{\varphi}\right)^2 \\
&\quad + \eta^{-1}\left(-\left(\frac{b}{2d}\right)'\left(c\tilde{\psi}+d\tilde{\varphi}\right)^2 + \left(\frac{c}{2d}\right)'\tilde{\psi}^2 - \left(\frac{1}{d}\right)'\tilde{\psi}\left(c\tilde{\psi}+d\tilde{\varphi}\right)\right) \\
&= \left\{\left(1-\eta^{-1}\left(\frac{b}{d}\right)'\right)cd - Q(x)ab - \eta^{-1}\left(\frac{1}{d}\right)'d\right\}\tilde{\psi}\tilde{\varphi} \\
&\quad + \frac{1}{2}\left\{\left(1-\eta^{-1}\left(\frac{b}{d}\right)'\right)c^2 - Q(x)a^2 + \eta^{-1}\left(\frac{c}{d}\right)' - 2\eta^{-1}\left(\frac{1}{d}\right)'c\right\}\tilde{\psi}^2 \\
&\quad + \frac{1}{2}\left\{\left(1-\eta^{-1}\left(\frac{b}{d}\right)'\right)d^2 - Q(x)b^2\right\}\tilde{\varphi}^2.
\end{aligned}$$

In order that $\widetilde{H}$ may be of Birkhoff normal form, it is sufficient that the following equalities should be satisfied:

$$(28) \qquad \left(1-\eta^{-1}\left(\frac{b}{d}\right)'\right)c^2 - Q(x)a^2 + \eta^{-1}\left(\frac{c}{d}\right)' - 2\eta^{-1}\left(\frac{1}{d}\right)'c = 0,$$

$$(29) \qquad \left(1-\eta^{-1}\left(\frac{b}{d}\right)'\right)d^2 - Q(x)b^2 = 0.$$

In particular, since

$$
\begin{aligned}
(29) \iff & \left(\frac{d}{b}\right)^2 - \eta^{-1}\left(\frac{b}{d}\right)'\left(\frac{d}{b}\right)^2 - Q(x) = 0 \\
\iff & \left(\frac{d}{b}\right)^2 + \eta^{-1}\left(\frac{d}{b}\right)' = Q(x),
\end{aligned}
$$

$\eta d/b$ satisfies the Riccati equation (6). Furthermore, since (24) implies $(b/d)' = (a/c)' - (1/cd)'$, we have

$$
\begin{aligned}
(28) \iff & \left(1 - \eta^{-1}\left(\frac{a}{c}\right)'\right)c^2 - Q(x)a^2 + \eta^{-1}\left\{\left(\frac{1}{cd}\right)'c^2 + \left(\frac{c}{d}\right)' - 2\left(\frac{1}{d}\right)'c\right\} = 0 \\
\iff & \left(1 - \eta^{-1}\left(\frac{a}{c}\right)'\right)c^2 - Q(x)a^2 = 0 \\
\iff & \left(\frac{c}{a}\right)^2 + \eta^{-1}\left(\frac{c}{a}\right)' = Q(x).
\end{aligned}
$$

Hence $\eta c/a$ also satisfies the Riccati equation (6). Note that the Riccati equation (6) can be solved in a singular-perturbative manner and we obtain two formal solutions $S_\pm$ given by (7). In this situation $\eta c/a$ and $\eta d/b$ must be different solutions since it follows from (24) that

$$
\left(\frac{d}{b}\right) - \left(\frac{c}{a}\right) = \frac{1}{ab}.
$$

Thus we may assume

$$
(30) \qquad \begin{cases} \eta\dfrac{c}{a} = S_{\mathrm{odd}} + S_{\mathrm{even}} = S_+ \\[2ex] \eta\dfrac{d}{b} = -S_{\mathrm{odd}} + S_{\mathrm{even}} = S_- \end{cases}
$$

and

$$
(31) \qquad \frac{1}{ab} = -2\eta^{-1}S_{\mathrm{odd}}.
$$

These relations (30) and (31) are describing the condition that the transformation (23) is canonical and reduces the original Hamiltonian system (13)–(14) into its Birkhoff normal form. By (27) and the identity

$$
\begin{aligned}
-\left(\frac{b}{d}\right)'cd - \left(\frac{1}{d}\right)'d &= \left(\frac{d}{b}\right)'\frac{b^2c}{d} + \frac{d'}{d} \\
&= \left(\frac{d}{b}\right)'\frac{b}{d}(ad-1) + \frac{d'}{d}
\end{aligned}
$$

$$
\begin{aligned}
&= \left(\frac{d}{b}\right)' ab - \left(\frac{d}{b}\right)' \frac{b}{d} + \frac{d'}{d} \\
&= \left(\frac{d}{b}\right)' ab + \frac{b'}{b},
\end{aligned}
$$

we find also that the coefficient of $\widetilde{\psi}\widetilde{\varphi}$ in the Birkhoff normal form is given by the following:

$$
\begin{aligned}
(32)\qquad & \left(1 - \eta^{-1}\left(\frac{b}{d}\right)'\right) cd - Q(x)ab - \eta^{-1}\left(\frac{1}{d}\right)' d \\
&= cd - Q(x)ab + \eta^{-1}\left(\frac{d}{b}\right)' ab + \eta^{-1}\frac{b'}{b} \\
&= cd - \left(\frac{d}{b}\right)^2 ab + \eta^{-1}\frac{b'}{b} \\
&= -\frac{d}{b} + \eta^{-1}\frac{b'}{b}.
\end{aligned}
$$

However, it is obvious that (30) and (31) cannot determine the transformation uniquely. Concerning the determination of $a$, $b$, $c$ and $d$ we have the following (typical) options:

**Idea A** : We determine $a$, $b$, $c$ and $d$ in such a way that the coefficient (32) of $\widetilde{\psi}\widetilde{\varphi}$ in the Birkhoff normal form may become as simple as possible. For that purpose we should define $b$ by solving

$$
(33)\qquad \frac{db}{dx} - \left(S_- + \eta\sqrt{Q(x)}\right) b = 0
$$

in view of (30) and (32). Consequently the Hamiltonian of the Birkhoff normal form becomes

$$
(34)\qquad \widetilde{H} = \sqrt{Q(x)}\widetilde{\psi}\widetilde{\varphi}.
$$

Note that due to the assumption that $b$ is a formal power series of $\eta^{-1}$ we cannot eliminate the coefficient of $\widetilde{\psi}\widetilde{\varphi}$ completely and the top degree part $\sqrt{Q(x)}$ remains. The differential equation (33) for $b$ together with (30) and (31) determines $a$, $b$, $c$ and $d$ modulo constants of integration.

In this determination of the transformation we have to solve the differential equation (33) and the transformation itself inevitably contains some constants of integration. In that sense this approach is closer to the construction of WKB solutions via eiconal and transport equations.

**Idea B** : To determine $a$, $b$, $c$ and $d$ we make the following additional requirement:

$$
(35)\qquad a = -b.
$$

The meaning of this requirement is to pick out the odd part of solutions as the coefficient of $\tilde{\psi}\tilde{\varphi}$ and the even part as the canonical transformation $a$ and $b$ (cf. (19)). As a matter of fact, (35) together with (31) entails

$$a = -b = \left(2\eta^{-1}S_{\mathrm{odd}}\right)^{-1/2}, \tag{36}$$

and further the coefficient of $\tilde{\psi}\tilde{\varphi}$ becomes

$$\begin{aligned} -\frac{d}{b} + \eta^{-1}\frac{b'}{b} &= -\eta^{-1}\left(-S_{\mathrm{odd}} + S_{\mathrm{even}}\right) + \eta^{-1}\frac{d}{dx}\log\left(2\eta^{-1}S_{\mathrm{odd}}\right)^{-1/2} \\ &= -\eta^{-1}\left(-S_{\mathrm{odd}} + S_{\mathrm{even}}\right) + \eta^{-1}\left(-\frac{1}{2}\right)\frac{d}{dx}\log S_{\mathrm{odd}} \\ &= \eta^{-1}S_{\mathrm{odd}} \end{aligned} \tag{37}$$

thanks to the relation (8). We thus obtain solutions of (4) of the form

$$\psi = \left(2\eta^{-1}S_{\mathrm{odd}}\right)^{-1/2}\left\{\alpha\exp\left(\int^x S_{\mathrm{odd}}dx\right) + \beta\exp\left(-\int^x S_{\mathrm{odd}}dx\right)\right\}. \tag{38}$$

The requirement (35) enables us to determine the transformation uniquely. This approach is closer to the construction of WKB solutions via the Riccati equation.

In this way the WKB solutions of Schrödinger equations can be constructed also by using reduction of Hamiltonian systems to Birkhoff normal form. In Section 3 we employ this idea to construct formal solutions of Painlevé equations. Again there we will encounter a similar problem of unique determination of canonical transformations as above. Throughout this report we follow mainly the line of "Idea B" even in the case of Painlevé equations.

# 3 Construction of formal solutions of Painlevé equations via reduction to Birkhoff normal form

In this section we consider the construction of 2-parameter formal solutions of Painlevé equations $(P_J)$ $(J = \mathrm{I}, \ldots, \mathrm{VI})$ with a large parameter $\eta$, which are tabulated in Table 1 below.

**Table 1**

$$(P_{\mathrm{I}}) \qquad \frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t).$$

$$(P_{\mathrm{II}}) \qquad \frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + c).$$

$$(P_{\rm III})\qquad \frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \eta^2\left[16c_\infty\lambda^3 + \frac{8c'_\infty\lambda^2}{t} - \frac{8c'_0}{t} - \frac{16c_0}{\lambda}\right].$$

$$(P_{\rm IV})\qquad \frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{2}{\lambda} + \eta^2\left[\frac{3}{2}\lambda^3 + 4t\lambda^2 + \left(2t^2 + 8c_1\right)\lambda - \frac{8c_0}{\lambda}\right].$$

$$(P_{\rm V})\qquad \frac{d^2\lambda}{dt^2} = \left(\frac{1}{2\lambda} + \frac{1}{\lambda-1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(2\lambda - \frac{1}{2\lambda}\right)$$
$$+\eta^2\frac{2\lambda(\lambda-1)^2}{t^2}\left[(c_0+c_\infty) - \frac{c_0}{\lambda^2} - \frac{c_2 t}{(\lambda-1)^2} - \frac{c_1 t^2(\lambda+1)}{(\lambda-1)^3}\right].$$

$$(P_{\rm VI})\qquad \frac{d^2\lambda}{dt^2} = \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}$$
$$+\frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[1 - \frac{\lambda^2 - 2t\lambda + t}{4\lambda^2(\lambda-1)^2}\right.$$
$$\left.+\eta^2\left\{(c_0+c_1+c_t+c_\infty) - \frac{c_0 t}{\lambda^2} + \frac{c_1(t-1)}{(\lambda-1)^2} - \frac{c_t t(t-1)}{(\lambda-t)^2}\right\}\right].$$

As is well known, Painlevé equations can also be represented in the form of Hamiltonian systems

$$(H_J)\qquad d\lambda/dt = \eta\partial K_J/\partial\nu,\qquad d\nu/dt = -\eta\partial K_J/\partial\lambda$$

(cf., e.g., [O]). One explicit choice of Hamiltonians $K_J(t,\lambda,\nu,\eta)$ is the following:

**Table 2**

$$K_{\rm I} = \frac{1}{2}\left[\nu^2 - \left(4\lambda^3 + 2t\lambda\right)\right].$$

$$K_{\rm II} = \frac{1}{2}\left[\nu^2 - \left(\lambda^4 + t\lambda^2 + 2c\lambda\right)\right].$$

$$K_{\rm III} = \frac{2\lambda^2}{t}\left[\nu^2 - \eta^{-1}\frac{3\nu}{2\lambda} - \left(\frac{c_0t^2}{\lambda^4} + \frac{c'_0 t}{\lambda^3} + \frac{c'_\infty t}{\lambda} + c_\infty t^2\right)\right].$$

$$K_{\rm IV} = 2\lambda\left[\nu^2 - \eta^{-1}\frac{\nu}{\lambda} - \left(\frac{c_0}{\lambda^2} + c_1 + \left(\frac{\lambda+2t}{4}\right)^2\right)\right].$$

$$K_{\rm V} = \frac{\lambda(\lambda-1)^2}{t}$$
$$\times\left[\nu^2 - \eta^{-1}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1}\right)\nu - \left(\frac{c_0}{\lambda^2} + \frac{c_1t^2}{(\lambda-1)^4} + \frac{c_2 t}{(\lambda-1)^3} + \frac{c_\infty}{(\lambda-1)^2}\right)\right].$$

$$K_{\rm VI} = \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)}$$
$$\times\left[\nu^2 - \eta^{-1}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1}\right)\nu - \left(\frac{c_0}{\lambda^2} + \frac{c_1}{(\lambda-1)^2} + \frac{c_\infty}{\lambda(\lambda-1)} + \frac{c_t}{(\lambda-t)^2}\right)\right].$$

In what follows we try to construct formal solutions of $(P_J)$ by using reduction of this Hamiltonian system $(H_J)$ to its Birkhoff normal form.

Let us first note that each Painlevé equation has the following structure in common:

$$\frac{d^2\lambda}{dt^2} = G_J\left(\lambda, \frac{d\lambda}{dt}, t\right) + \eta^2 F_J(\lambda, t), \tag{39}$$

where $F_J$ and $G_J$ are rational functions. In view of (39) we easily find that $(P_J)$ has the following formal power series solutions denoted by $\lambda_J^{(0)}(t)$:

$$\lambda_J^{(0)}(t) = \lambda_0(t) + \eta^{-2}\lambda_2(t) + \eta^{-4}\lambda_4(t) + \cdots, \tag{40}$$

where the top term $\lambda_0(t)$ satisfies

$$F_J(\lambda_0(t), t) = 0$$

and the other $\lambda_{2j}(t)$ $(j \geq 1)$ are determined in a recursive manner. Corresponding to these solutions (40), there exist formal power series solutions called 0-parameter solutions of $(H_J)$:

$$\begin{cases} \lambda_J^{(0)}(t) &= \lambda_0(t) + \eta^{-2}\lambda_2(t) + \eta^{-4}\lambda_4(t) + \cdots \\ \nu_J^{(0)}(t) &= \eta^{-1}\nu_1(t) + \eta^{-3}\nu_3(t) + \eta^{-5}\nu_5(t) + \cdots \end{cases}$$

(cf. [KT1, Proposition 1.1]). Let us next consider the following localization of $(H_J)$ at this 0-parameter solution:

$$\lambda = \lambda_J^{(0)}(t) + \eta^{-1/2}U, \qquad \nu = \nu_J^{(0)}(t) + \eta^{-1/2}V, \tag{41}$$

that is, we transform the unknown function of $(H_J)$ from $(\lambda, \nu)$ to $(U, V)$. Then we readily verify that $(U, V)$ must obey another Hamiltonian system

$$dU/dt = \eta\partial\mathcal{K}_J/\partial V, \qquad dV/dt = -\eta\partial\mathcal{K}_J/\partial U, \tag{42}$$

where $\mathcal{K}_J$ is given by the following:

$$\mathcal{K}_J = \sum_{j+k\geq 2} \eta^{-(j+k-2)/2} \frac{1}{j!k!} \frac{\partial^{j+k} K_J}{\partial\lambda^j \partial\nu^k} \left(t, \lambda_J^{(0)}(t), \nu_J^{(0)}(t), \eta\right) U^j V^k. \tag{43}$$

Now the main result of this report is the following:

**Theorem 1** *There exists a formal canonical transformation* $(U, V) \longmapsto (\tilde{U}, \tilde{V})$ *of the form*

$$\begin{cases} U &= u_0(\tilde{U}, \tilde{V}) + \eta^{-1/2}u_1(\tilde{U}, \tilde{V}) + \cdots, \\ V &= v_0(\tilde{U}, \tilde{V}) + \eta^{-1/2}v_1(\tilde{U}, \tilde{V}) + \cdots, \end{cases} \tag{44}$$

*where $u_j$ and $v_j$ are homogeneous polynomials of degree $(j+1)$ in $(\tilde{U},\tilde{V})$ (whose coefficients are formal power series of $\eta^{-1/2}$ with coefficients being functions of $t$), so that the Hamiltonian system (42) may be taken into the following normal form:*

$$d\tilde{U}/dt = \eta\partial\tilde{\mathcal{K}}_J/\partial\tilde{V}, \qquad d\tilde{V}/dt = -\eta\partial\tilde{\mathcal{K}}_J/\partial\tilde{U}, \tag{45}$$

*where*

$$\tilde{\mathcal{K}}_J = \sum_{l=0}^{\infty}\eta^{-l}f^{(l)}(t,\eta)\left(\tilde{U}\tilde{V}\right)^{l+1} \tag{46}$$

*and each $f^{(l)}(t,\eta) = \sum_{j\geq 0}\eta^{-j/2}f^{(l)}_{j/2}(t)$ is a formal power series of $\eta^{-1/2}$ with coefficients being functions of $t$.*

*Remark* The concrete form of the first few terms of $f^{(l)}$ in the case of $J = \mathrm{I}$ is the following:

$$\begin{aligned} f^{(0)} &= (12\lambda_0)^{1/2} - \eta^{-2}\frac{3^2\cdot 5^2}{2}(12\lambda_0)^{-9/2} + \cdots \\ f^{(1)} &= 15\,(12\lambda_0)^{-2} + \eta^{-2}3^3\cdot 5^2\cdot 31\,(12\lambda_0)^{-7} + \cdots \\ f^{(2)} &= -3\cdot 5\cdot 47\,(12\lambda_0)^{-9/2} + \cdots. \end{aligned}$$

For the top degree part $f_0^{(0)}(t)$ we also have the following equalities for any $J$:

$$f_0^{(0)}(t) = \sqrt{\frac{\partial F_J}{\partial\lambda}(\lambda_0(t),t)}. \tag{47}$$

Theorem 1 claims that the Hamiltonian system (42) can be transformed into its Birkhoff normal form. Since the reduced Hamiltonian $\tilde{\mathcal{K}}_J$ depends only on the product $\tilde{U}\tilde{V}$, the system (45) is easily solved; taking account of the fact that the product $\tilde{U}\tilde{V}$ is independent of $t$, we find

$$\left\{\begin{aligned} \tilde{U} &= \alpha\exp\left(\eta\int^t\sum\eta^{-l}(l+1)f^{(l)}(s,\eta)(-\alpha\beta)^l ds\right) \\ \tilde{V} &= -\beta\exp\left(-\eta\int^t\sum\eta^{-l}(l+1)f^{(l)}(s,\eta)(-\alpha\beta)^l ds\right) \end{aligned}\right. \tag{48}$$

gives a solution of (45). Substituting (48) into (44) and then into (41), we obtain 2-parameter formal solutions of $(H_J)$ and $(P_J)$.

Let us now sketch the proof of Theorem 1. The proof consists of the following two steps; reduction of the linear part and that of the nonlinear part. We first consider reduction of the linear part, that is, we seek for a linear canonical transformation

$$\left\{\begin{aligned} U &= a(t,\eta)\tilde{U} + b(t,\eta)\tilde{V} \\ V &= c(t,\eta)\tilde{U} + d(t,\eta)\tilde{V} \end{aligned}\right. \tag{49}$$

with the generating function

$$W(t,\tilde{U},V) = -\frac{b}{2d}V^2 + \frac{c}{2d}\tilde{U}^2 - \frac{1}{d}\tilde{U}V \tag{50}$$

which transforms the Hamiltonian system (42) into its Birkhoff normal form up to quadratic terms. By (49) the Hamiltonian $\mathcal{K}_J$ is transformed into

(51)

$$\begin{aligned}
\tilde{\mathcal{K}}_J &= \mathcal{K}_J + \eta^{-1}\frac{\partial W}{\partial t} \\
&= \frac{1}{2}\frac{\partial^2 K}{\partial \lambda^2}\left(a\tilde{U}+b\tilde{V}\right)^2 + \frac{\partial^2 K}{\partial\lambda\partial\nu}\left(a\tilde{U}+b\tilde{V}\right)\left(c\tilde{U}+d\tilde{V}\right) + \frac{1}{2}\frac{\partial^2 K}{\partial\nu^2}\left(c\tilde{U}+d\tilde{V}\right)^2 \\
&\quad + \eta^{-1}\left\{-\left(\frac{b}{2d}\right)'\left(c\tilde{U}+d\tilde{V}\right)^2 + \left(\frac{c}{2d}\right)'\tilde{U}^2 - \left(\frac{1}{d}\right)'\tilde{U}\left(c\tilde{U}+d\tilde{V}\right)\right\} \\
&\quad + (\text{terms of degree greater than 2 in } (\tilde{U},\tilde{V})).
\end{aligned}$$

(Here and in what follows we often omit the suffix $J$ for simplicity and abbreviate $(\partial^2 K_J/\partial\lambda^2)(t,\lambda_J^{(0)}(t),\nu_J^{(0)}(t),\eta)$ to $\partial^2 K/\partial\lambda^2$ etc. if there is no fear of confusions.) Namely

$$\begin{aligned}
(\text{coeff. of } \tilde{U}\tilde{V}) &= \frac{\partial^2 K}{\partial\lambda^2}ab + \frac{\partial^2 K}{\partial\lambda\partial\nu}(ad+bc) + \frac{\partial^2 K}{\partial\nu^2}cd \\
&\quad - \eta^{-1}\left(\left(\frac{b}{d}\right)' cd + \left(\frac{1}{d}\right)' d\right),
\end{aligned} \tag{52}$$

$$\begin{aligned}
(\text{coeff. of } \tilde{U}^2) &= \frac{1}{2}\frac{\partial^2 K}{\partial\lambda^2}a^2 + \frac{\partial^2 K}{\partial\lambda\partial\nu}ac + \frac{1}{2}\frac{\partial^2 K}{\partial\nu^2}c^2 \\
&\quad + \eta^{-1}\left(-\left(\frac{b}{2d}\right)' c^2 + \left(\frac{c}{2d}\right)' - \left(\frac{1}{d}\right)' c\right),
\end{aligned} \tag{53}$$

$$(\text{coeff. of } \tilde{V}^2) = \frac{1}{2}\frac{\partial^2 K}{\partial\lambda^2}b^2 + \frac{\partial^2 K}{\partial\lambda\partial\nu}bd + \frac{1}{2}\frac{\partial^2 K}{\partial\nu^2}d^2 - \eta^{-1}\left(\frac{b}{2d}\right)' d^2. \tag{54}$$

We are thus required to choose $a$, $b$, $c$ and $d$ so that (53) and (54) may vanish. It is really possible, that is, we can prove

**Proposition 1** *There exist $a$, $b$, $c$ and $d$ which satisfy*

$$ad - bc = 1, \tag{55}$$

$$\textit{(coeff. of } \tilde{U}^2\textit{)} = 0, \tag{56}$$

$$\textit{(coeff. of } \tilde{V}^2\textit{)} = 0 \tag{57}$$

*together with the additional requirement*

$$a = -b. \tag{58}$$

*These conditions (55)–(58) determine a, b, c and d (almost) uniquely. Furthermore (55)–(58) entail the following:*

$$\text{(coeff. of } \tilde{U}\tilde{V}) = \eta^{-1} S_{\text{odd}}, \tag{59}$$

*where $S_{\text{odd}}$ denotes the odd part (in the sense of [AKT2, Definition 2.1]) of solutions of the Riccati equation associated with the Fréchet derivative (i.e., linearized equation) of $(H_J)$ along the 0-parameter solution $(\lambda_J^{(0)}, \nu_J^{(0)})$.*

Before mentioning some comments on Proposition 1, let us recall here the definition of the *Riccati equation associated with the Fréchet derivative of $(H_J)$*.

Substituting $\lambda = \lambda_J^{(0)} + \psi$ and $\nu = \nu_J^{(0)} + \varphi$ into $(H_J)$, we find that the Fréchet derivative of $(H_J)$ is given by the following:

$$\begin{cases} \psi' &= \eta \left( \dfrac{\partial^2 K}{\partial \lambda \partial \nu} \psi + \dfrac{\partial^2 K}{\partial \nu^2} \varphi \right), \\ \varphi' &= -\eta \left( \dfrac{\partial^2 K}{\partial \lambda^2} \psi + \dfrac{\partial^2 K}{\partial \lambda \partial \nu} \varphi \right). \end{cases} \tag{60}$$

We consider WKB solutions of (60), which is of the form

$$\psi = \exp \int^t S dt, \qquad \varphi = \exp \int^t T dt.$$

Then $S$ and $T$ must satisfy

$$\left( S - \eta \frac{\partial^2 K}{\partial \lambda \partial \nu} \right) \exp \int^t S dt - \eta \frac{\partial^2 K}{\partial \nu^2} \exp \int^t T dt = 0, \tag{61}$$

$$\eta \frac{\partial^2 K}{\partial \lambda^2} \exp \int^t S dt + \left( T + \eta \frac{\partial^2 K}{\partial \lambda \partial \nu} \right) \exp \int^t T dt = 0. \tag{62}$$

Let us take the logarithmic derivative of (61).

$$\frac{d}{dt} \log \left( S - \eta \frac{\partial^2 K}{\partial \lambda \partial \nu} \right) + S = \frac{d}{dt} \log \frac{\partial^2 K}{\partial \nu^2} + T. \tag{63}$$

Furthermore, since neither $\exp \int^t S dt$ nor $\exp \int^t T dt$ is equal to zero, (61) and (62) entail

$$\left( S - \eta \frac{\partial^2 K}{\partial \lambda \partial \nu} \right) \left( T + \eta \frac{\partial^2 K}{\partial \lambda \partial \nu} \right) + \eta^2 \frac{\partial^2 K}{\partial \lambda^2} \frac{\partial^2 K}{\partial \nu^2} = 0. \tag{64}$$

A single equation which determines $S$ can be easily obtained from (63) and (64). In fact, putting

$$S^\dagger = S - \eta\frac{\partial^2 K}{\partial\lambda\partial\nu}, \qquad T^\dagger = T + \eta\frac{\partial^2 K}{\partial\lambda\partial\nu},$$

we have

$$\begin{aligned} T^\dagger &= S^\dagger + \frac{d}{dt}\log S^\dagger + 2\eta\frac{\partial^2 K}{\partial\lambda\partial\nu} - \frac{d}{dt}\log\frac{\partial^2 K}{\partial\nu^2}, \\ S^\dagger T^\dagger &+ \eta^2\frac{\partial^2 K}{\partial\lambda^2}\frac{\partial^2 K}{\partial\nu^2} = 0. \end{aligned}$$

Hence

$$(S^\dagger)^2 + \frac{dS^\dagger}{dt} + \left(2\eta\frac{\partial^2 K}{\partial\lambda\partial\nu} - \frac{d}{dt}\log\frac{\partial^2 K}{\partial\nu^2}\right)S^\dagger + \eta^2\frac{\partial^2 K}{\partial\lambda^2}\frac{\partial^2 K}{\partial\nu^2} = 0,$$

or, in terms of the original $S$ instead of $S^\dagger$,

$$\begin{aligned} S^2 + \frac{dS}{dt} - \frac{d}{dt}\log\frac{\partial^2 K}{\partial\nu^2} + \eta^2\left(\frac{\partial^2 K}{\partial\lambda^2}\frac{\partial^2 K}{\partial\nu^2} - \left(\frac{\partial^2 K}{\partial\lambda\partial\nu}\right)^2\right) \\ + \eta\left(\frac{\partial^2 K}{\partial\lambda\partial\nu}\frac{d}{dt}\log\frac{\partial^2 K}{\partial\nu^2} - \frac{d}{dt}\frac{\partial^2 K}{\partial\lambda\partial\nu}\right) = 0 \end{aligned} \tag{65}$$

should be satisfied. This is the Riccati equation associated with the Fréchet derivative of $(H_J)$.

Just like (6) we can solve (65) in a singular-perturbative manner to obtain two formal power series solutions

$$\begin{aligned} S_\pm &= \pm\eta S_{-1}(t) + S_{\pm,0}(t) + \eta^{-1}S_{\pm,1}(t) + \cdots, \\ &= \pm S_{\mathrm{odd}} + S_{\mathrm{even}}. \end{aligned} \tag{66}$$

By comparing the odd part (in the sense of [AKT2, Definition2.1]) of (65) we find that, instead of (8), the following relation holds in the present situation:

$$S_{\mathrm{even}} = \frac{1}{2}\frac{d}{dt}\left(\log\frac{\partial^2 K}{\partial\nu^2} - \log S_{\mathrm{odd}}\right). \tag{67}$$

Furthermore, since the degree 0 part (in $\eta$) of $(\partial^2 K/\partial\lambda\partial\nu)$ vanishes, by straightforward computations we can show the following:

$$S_{-1}(t) = \left(\frac{\partial F_J}{\partial\lambda}(\lambda_0, t)\right)^{1/2} \tag{68}$$

(cf. (1.11) in [KT1], (1.35) and (1.41) in [KT2]). The relation (47) is an immediate consequence of (59) and (68).

Proposition 1 can be proved by a similar argument as that in Section 2 which verified the Hamiltonian system (13)–(14) is reduced into its Birkhoff normal form. See [T3] for the details of the proof of Proposition 1. Here we only note that the additional requirement (58) again determines a canonical transformation (49) (almost) uniquely. Its coefficients $a$, $b$, $c$ and $d$ are, as a matter of fact, given by the following:

$$a = -b = \left(\frac{\partial^2 K}{\partial \nu^2}\frac{1}{2\eta^{-1}S_{\mathrm{odd}}}\right)^{1/2}, \tag{69}$$

$$c = \left(2\eta^{-1}S_{\mathrm{odd}}\frac{\partial^2 K}{\partial \nu^2}\right)^{-1/2}\left(\eta^{-1}S_{+} - \frac{\partial^2 K}{\partial\lambda\partial\nu}\right), \tag{70}$$

$$d = -\left(2\eta^{-1}S_{\mathrm{odd}}\frac{\partial^2 K}{\partial \nu^2}\right)^{-1/2}\left(\eta^{-1}S_{-} - \frac{\partial^2 K}{\partial\lambda\partial\nu}\right). \tag{71}$$

By this reduction of the linear part we thus obtain the following reduced Hamiltonian:

(72)

$$\begin{aligned}\mathcal{K}_J &= \eta^{-1}S_{\mathrm{odd}}UV + \sum_{j+k\geq 3}\eta^{-(j+k-2)/2}\frac{1}{j!k!}\frac{\partial^{j+k}K}{\partial\lambda^j\partial\nu^k}(aU+bV)^j(cU+dV)^k \\ &= \eta^{-1}S_{\mathrm{odd}}UV + \sum_{|\vec{j}+\vec{k}|\geq 3}\eta^{-(|\vec{j}+\vec{k}|-2)/2}\frac{1}{\vec{j}!\vec{k}!}\frac{\partial^{|\vec{j}+\vec{k}|}K}{\partial\lambda^{|\vec{j}|}\partial\nu^{|\vec{k}|}}a^{j_1}b^{j_2}c^{k_1}d^{k_2}U^{j_1+k_1}V^{j_2+k_2},\end{aligned}$$

where $\vec{j} = (j_1, j_2)$ and $\vec{k} = (k_1, k_2)$. (We have omitted tildes ($\tilde{\ }$) for the sake of simplicity.) The Hamiltonian (72) is written also in the following form:

$$\mathcal{K}_J = \eta^{-1}S_{\mathrm{odd}}UV + \sum_{\substack{p+q\geq 1\\ p,q\geq -1}}\eta^{-(p+q)/2}K_{pq}(t,\eta)U^{p+1}V^{q+1} \tag{73}$$

where

$$K_{pq}(t,\eta) = \sum_{\substack{j_1+k_1=p+1\\ j_2+k_2=q+1}}\frac{1}{\vec{j}!\vec{k}!}\frac{\partial^{|\vec{j}+\vec{k}|}K}{\partial\lambda^{|\vec{j}|}\partial\nu^{|\vec{k}|}}a^{j_1}b^{j_2}c^{k_1}d^{k_2}. \tag{74}$$

What we next have to do is to find reduction of the nonlinear part, that is, a canonical transformation with the trivial linear terms

$$\left\{\begin{aligned} U &= \tilde{U} + \eta^{-1/2}u_1(t,\tilde{U},\tilde{V},\eta) + \eta^{-1}u_2(t,\tilde{U},\tilde{V},\eta) + \cdots \\ V &= \tilde{V} + \eta^{-1/2}v_1(t,\tilde{U},\tilde{V},\eta) + \eta^{-1}v_2(t,\tilde{U},\tilde{V},\eta) + \cdots,\end{aligned}\right. \tag{75}$$

where

$$u_j(t,\widetilde{U},\widetilde{V},\eta) = \sum_{\substack{p+q=j-1\\ p,q\geq -1}} u_{pq}(t,\eta)\widetilde{U}^{p+1}\widetilde{V}^{q+1} \tag{76}$$

$$v_j(t,\widetilde{U},\widetilde{V},\eta) = \sum_{\substack{p+q=j-1\\ p,q\geq -1}} v_{pq}(t,\eta)\widetilde{U}^{p+1}\widetilde{V}^{q+1} \tag{77}$$

with $u_{pq}$ and $v_{pq}$ being formal power series of $\eta^{-1/2}$, which transforms the Hamiltonian $\mathcal{K}_J$ into its Birkhoff normal form. For that purpose we again make use of a generating function of the following form:

$$\begin{aligned} W &= W(t,\widetilde{U},V) \\ &= -\widetilde{U}V + \sum_{\substack{p+q\geq 1\\ p,q\geq -1}} \eta^{-(p+q)/2} a_{pq}(t,\eta)\widetilde{U}^{p+1}V^{q+1}. \end{aligned} \tag{78}$$

Roughly speaking, by introducing more additional requirements, we can uniquely determine $\{a_{pq}\}$ in a recursive manner so that the associated canonical transformation

$$\begin{cases} U = -\dfrac{\partial W}{\partial V} = \widetilde{U} - \displaystyle\sum_{\substack{p+q\geq 1\\ p,q\geq -1}} \eta^{-(p+q)/2}(q+1)a_{pq}\widetilde{U}^{p+1}V^{q} \\ \widetilde{V} = -\dfrac{\partial W}{\partial \widetilde{U}} = V - \displaystyle\sum_{\substack{p+q\geq 1\\ p,q\geq -1}} \eta^{-(p+q)/2}(p+1)a_{pq}\widetilde{U}^{p}V^{q+1} \end{cases} \tag{79}$$

reduces the Hamiltonian (73) into its Birkhoff normal form. Note that, if we successfully find such $\{a_{pq}\}$, then $u_j(t,\widetilde{U},\widetilde{V},\eta)$ and $v_j(t,\widetilde{U},\widetilde{V},\eta)$ are explicitly given by the following:

$$u_j = -\sum_{\substack{p+q+\mu_1+\cdots+\mu_k=j\\ p+q\geq 1, p,q\geq -1,\mu_l\geq 0}} (q+1)a_{pq}\widetilde{U}^{p+1}v_{\mu_1}\cdots v_{\mu_k} \tag{80}$$

$$v_j = \sum_{\substack{p+q+\mu_1+\cdots+\mu_{k+1}=j\\ p+q\geq 1, p,q\geq -1,\mu_l\geq 0}} (p+1)a_{pq}\widetilde{U}^{p}v_{\mu_1}\cdots v_{\mu_{k+1}} \tag{81}$$

($j=1,2,3,\ldots$) where $u_0$ and $v_0$ respectively denote $\widetilde{U}$ and $\widetilde{V}$. (The relations (80) and (81) recursively determine $\{u_j\}$ and $\{v_j\}$ from $\{a_{pq}\}$.) We omit the details of the argument here and only refer the reader to [T3].

# 4 Local behavior of formal solutions of Painlevé equations near regular-type singular points

We have seen in the preceding section that singular-perturbative reduction of $(H_J)$ (more precisely, its localization at the 0-parameter solution) to Birkhoff normal form produces 2-parameter formal solutions of $(P_J)$. In this section we study their local behavior at fixed regular-type singular points. As a typical example of fixed regular-type singular points of Painlevé equations we pick up the origin $t = 0$ of the sixth Painlevé equation $(P_{\mathrm{VI}})$ and discuss the problem only for this typical example in this report.

As is shown in Theorem 2 below, the regular-type singularness of fixed singular points of $(P_J)$ ($t = 0$ of $(P_{\mathrm{VI}})$ here) should entail the simpleness of poles which the coefficients $f^{(l)}(t, \eta)$ of the Birkhoff normal form may possess there. Furthermore, in the global study of $(P_J)$ the residues of $f^{(l)}(t, \eta)$ at regular-type singular points would play an important role. Hence it is desirable to be able to compute such residues explicitly. However, our choice of Hamiltonians $K_J(t, \lambda, \nu, \eta)$ which is listed up in Table 2 is not convenient for that purpose; if we work with $K_J$, we can show the simpleness of poles, but the computation of the residues becomes quite difficult. To overcome this difficulty we use the following "polynomial Hamiltonian $H_{\mathrm{VI}}$"

(82)
$$d\lambda/dt = \eta \partial H_{\mathrm{VI}}/\partial \mu, \qquad d\mu/dt = -\eta \partial H_{\mathrm{VI}}/\partial \lambda$$

where

(83)
$$\begin{aligned} H_{\mathrm{VI}} &= \frac{1}{t(t-1)} \Big[\lambda(\lambda-1)(\lambda-t)\mu^2 - \eta^{-1}\left\{\kappa_0(\lambda-1)(\lambda-t) + \kappa_1\lambda(\lambda-t)\right. \\ &\quad \left. + (\kappa_t - 1)\lambda(\lambda-1)\right\}\mu + \frac{1}{4}\eta^{-2}\left\{(\kappa_0+\kappa_1+\kappa_t-1)^2 - \kappa_\infty^2\right\}(\lambda - t)\Big], \end{aligned}$$

which is first discovered by Okamoto ([O]), instead of $K_{\mathrm{VI}}$ in this report. The relations between $H_{\mathrm{VI}}$, $\mu$, $\kappa_*$ and $K_{\mathrm{VI}}$, $\nu$, $c_*$ are given by the following:

$$\begin{aligned} &\frac{1}{4}(\kappa_*^2 - 1) = c_*\eta^2 \qquad \text{where } * = 0, 1, t, \\ &\frac{1}{4}(\kappa_\infty^2 - \kappa_0^2 - \kappa_1^2 - \kappa_t^2 - 1) = c_\infty \eta^2, \\ &\mu + \frac{1}{2}\eta^{-1}\left(\frac{1-\kappa_0}{\lambda} + \frac{1-\kappa_1}{\lambda-1} + \frac{1-\kappa_t}{\lambda-t}\right) = \nu, \\ &H_{\mathrm{VI}} + \frac{1}{2}\eta^{-2}(1-\kappa_t)\left(\frac{1-\kappa_0}{t} + \frac{1-\kappa_1}{t-1} + \frac{1}{\lambda-t}\right) = K_{\mathrm{VI}}. \end{aligned}$$

Note that every $\kappa_*$ ($* = 0, 1, t, \infty$) is a quantity of degree 1 in $\eta$.

Let us now state our results. The top degree part $\lambda_0(t)$ of our formal solutions is characterized by the equation $F_{\rm VI}(\lambda_0(t),t)=0$, i.e.,

$$(c_0+c_1+c_t+c_\infty)-c_0\frac{t}{\lambda_0^2}+c_1\frac{t-1}{(\lambda_0-1)^2}-c_t\frac{t(t-1)}{(\lambda_0-t)^2}=0.$$

This algebraic equation has six solutions, one of which shows the following behavior at $t=0$:

$$\lambda_0(t)=at+bt^2+\cdots \qquad \text{with } a=\frac{\sqrt{c_0}}{\sqrt{c_0}+\sqrt{c_t}}. \tag{84}$$

We restrict ourselves to this special choice of $\lambda_0(t)$ in this report. (The other cases will be discussed elsewhere.) Then, for 2-parameter formal solutions with the above top degree part $\lambda_0(t)$, we can verify the following:

**Theorem 2** *Let $f^{(l)}(t,\eta)$ be the coefficients of the Birkhoff normal form obtained in Theorem 1 from the localization of $(H_{\rm VI})$ at the 0-parameter solution with the top degree part $\lambda_0$ satisfying (84). Then each $f^{(l)}(t,\eta)$ has a simple pole at $t=0$ and*

$$\operatorname*{Res}_{t=0} f^{(0)}(t,\eta) = \eta^{-1}(\kappa_0+\kappa_t), \tag{85}$$

$$\operatorname*{Res}_{t=0} f^{(1)}(t,\eta) = 1, \tag{86}$$

$$\operatorname*{Res}_{t=0} f^{(l)}(t,\eta) = 0 \qquad (l\geq 2). \tag{87}$$

Furthermore, concerning the local behavior of the canonical transformation obtained in Section 3 which reduces the Hamiltonian system (82) to its Birkhoff normal form, we can also verify the following: (Note that, if we replace $K_{\rm VI}$ and $\nu$ by $H_{\rm VI}$ and $\mu$ respectively, all formulas in Section 3 hold even for the polynomial Hamiltonian $H_{\rm VI}$.)

**Proposition 2** *(i) The coefficients $a$, $b$, $c$ and $d$ of the linear part of the canonical transformation obtained in Proposition 1 (cf. (69)–(71) also) show the following local behavior at $t=0$:*

$$a = -b = \left(\frac{-\kappa_0\kappa_t}{\eta^{-1}(\kappa_0+\kappa_t)^3}\right)^{1/2} t+\cdots, \tag{88}$$

$$c = \left(\frac{\eta^{-1}(\kappa_0+\kappa_t)^3}{-\kappa_0\kappa_t}\right)^{1/2}\frac{1}{t}+\cdots, \tag{89}$$

$$d = 0\cdot\frac{1}{t}+\cdots. \tag{90}$$

*(ii) The coefficients $\{u_{pq}\}$ and $\{v_{pq}\}$ of the nonlinear part of the canonical transformation given respectively by (76) and (77) as well as the coefficients $\{a_{pq}\}$ of*

*the generating function (78) are holomorphic (more precisely, formal power series of $\eta^{-1/2}$ with holomorphic coefficients) at $t = 0$ for any $p, q$ with $p, q \geq -1$ and $p + q \geq 1$ (cf. (80) and (81)).*

For the proof of Theorem 2 and Proposition 2 see [T3].

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